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Mathematics and Poetry

From Boole (born 1815) to Whitehead, of present times, the development of mathematical logic has culminated in the assignment of vastly monumental meanings to the simplest of words, as "if", "then", "is", "or", "nor", "not", and others. (See the works of Whitehead, Russell, Keyser, Quine).

Probably "if", in a sense, is the progenitor of all the rest. *If* a triangle, or a circle; *if* a cone, or a pyramid, or a cylinder; *then* are quite all the truths of Euclid.

But, Bertrand Russell said "Mathematics is the science in which one never knows what one is talking about nor whether what one says is true". *And* all other competently informed minds agree with Russell.

Thus, through the vestibule of *some* "if" the mathematician passes into as pure a dream world as that of any poet. Indeed, poet and mathematician are, at times, singularly alike in their sense of detachment from so-called reality, and in their devotion, the one to his logic-spun domain, the other to his empire of fancy and emotion. Alongside of the poet's dream of a Paradise Lost, or an Ancient Mariner, may be set the mathematician's vision of a geometry not Euclid's or a physics not Newtonian.

Then, there are other characteristics, common to the truly poetical and truly mathematical natures. In each is the instinct for seeing simplicities in the complex thing, the mathematician reducing it to his points, lines, variables, the poet expressing it in terms of equally elementary things, as light, setting, beauty. Again, in each is the urge for a picturization with universal appeal. Tennyson's "Break, break on thy cold grey stones, O Sea" matches in the wide sweep of its call to human emotion the universality of reason's conviction that, given Euclid's assumptions, the sum of the angles of every plane triangle is 180 degrees.

S. T. Sanders.

Characteristic Functions in Statistics*

By J. F. Kenney
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9. *Moments and cumulants of certain distributions.* Notable abbreviations in certain demonstrations are made possible by the technique of the preceding sections. We shall study four important special distributions by means of this technique.

a. *Normal.* From (21) and Example 5, we have

$$K(w,x)=w^2\sigma^2/2. \tag{31}$$

A comparison of (31) with (22) shows that

$$\kappa_2=\sigma^2, \quad \kappa_r=0 \quad \text{for} \quad r\neq 2. \tag{32}$$

As a consequence of the uniqueness axiom of §4 these results completely characterize a normal distribution (except that $\kappa_1=0$ merely means that the origin of x is at ν_1). From (25) and (32) we obtain $\mu_3=0$ and $\mu_4=3\mu_2^2$.

b. *Binomial.* From (16) and (21) we have

$$K(w,x)=n\log(q+pe^w). \tag{33}$$

Applying (26) to (33) and then employing (25) we can obtain, in terms of n and p, as many of the cumulants and moments as may be desired. Thus we readily find

$$\kappa_1=\nu_1=np, \quad \kappa_2=\mu_2=npq, \quad \kappa_3=\mu_3=npq(q-p),$$
$$\kappa_4=npq\{(p-q)^2-2pq\}, \quad \mu_4=npq(1-6pq+3npq).$$

In applications of the Bernoulli theory dealing with ratios, the variable x/n is encountered. Let $x'=x/n$. Relations (12a) and (28) show that the rth moment or cumulant of x' can be obtained from the corresponding cumulant of x simply by multiplying the latter by $1/n^r$.

c. *Poisson.* A variate x is said to have a Poisson distribution (§7) if

$$f(x)=\frac{m^xe^{-m}}{x!}, \quad x=0,1,2,\cdots, \tag{34}$$

*This is a continuation of the first installment which appeared last month.—Editor.

where $\sum_x f(x)=1$, and m is the parameter of the distribution. Its characteristic function is

$$M(w,x)=e^{-m}\sum_x \frac{(me^w)^x}{x!}=e^{m(e^w-1)} \tag{35}$$

and its cumulant characteristic function is

$$\begin{aligned} K(w,x)&=m(e^w-1) \\ &=m(w+w^2/2!+w^3/3!+w^4/4!+\cdots). \end{aligned} \tag{36}$$

Hence,

$$\kappa_r=m,\ r=1,\ 2,\ 3,\cdots, \tag{37}$$

which shows that each cumulant of a Poisson distribution has the value of the mean. For the moments we turn to relation (25) and obtain

$$\nu_1=m=\mu_2=\mu_3, \quad \mu_4=3m^2+m.$$

d. *Pearson type III.* A continuous variate x is said to have a Pearson type III distribution with parameters b and h, $(b>0,\ h>0)$, if*

$$df=Cx^{b-1}e^{-hx}dx, \quad x\geqq 0. \tag{38}$$

Integrating (38) in the special case when $C=1=h$, we have the *gamma function*[18]

$$\int_0^\infty x^{b-1}e^{-x}dx=\Gamma(b),$$

and the *incomplete gamma function*

$$\int_0^x x^{b-1}e^{-x}dx.$$

More generally, the factor C is a function of b and h such that

$$M(0,x)=\int_0^\infty f(x)dx=1$$

*It is desirable to write the distribution function of a continuous variable in the form $f(x)dx$, which we denote by df, because if any transformation is made in x it must also be made in the differential dx.

in accord with (2). To determine C under this condition, let $u = hx$. Then

$$M(0,w) = Ch^{-b}\int_0^\infty u^{b-1}e^{-u}du = 1.$$

Hence, $$C = h^b/\Gamma(b)$$

since, upon replacing the dummy variable of integration x by u in the definition of the gamma function, we have

$$\int_0^\infty u^{b-1}e^{-u}du = \Gamma(b).$$

Then from (4) the characteristic function of (38) is given by

$$M(w,x) = \frac{h^b}{\Gamma(b)}\int_0^\infty x^{b-1}e^{-x(h-w)}dx.$$

This is transformed into

$$C(h-w)^{-b}\int_0^\infty u^{b-1}e^{-u}du$$

by the substitution $u = x(h-w)$. Whence we obtain

(39) $$M(w,x) = (1-w/h)^{-b}.$$

Consequently, the cumulant characteristic function is

(40) $$\begin{aligned} K(w,x) &= -b\log(1-w/h) \\ &= b(w/h + w^2/2h^2 + w^3/3h^3 + w^4/4h^4 + \cdots). \end{aligned}$$

Comparison of this result with (22) shows that the rth cumulant is

(41) $$\kappa_r = (r-1)!\, b\, h^{-r}.$$

The values of b and h are found by solving the simultaneous equations obtained from κ_r when $r = 1$ and $r = 2$. Thus we find

(42) $$b = \nu_1^2/\sigma^2, \quad h = \nu_1/\sigma^2.$$

If we define $\alpha_r = \mu_r/\sigma^r$ we obtain

$$\alpha_3 = 2b^{-\frac{1}{2}}, \quad \alpha_4 = 3 + 6b^{-1}.$$

These values satisfy the relation

$$2\alpha_4 - 3\alpha_3^2 - 6 = 0$$

which is known as the "criterion" for judging in advance whether a type III curve is appropriate for representing a proposed frequency distribution.

When $b=n/2$, $h=1/2$, and x is replaced by χ^2, (38) becomes

$$C(\chi^2)^{(n-2)/2}e^{-\chi^2/2}d(\chi^2) \tag{43}$$

where $1/C=2^{n/2}\Gamma(n/2)$. This is known as a chi-square distribution with n degrees of freedom. An important property of the chi-square distribution is that the *sum of any number of variates each having a chi-square distribution is itself distributed in a chi-square distribution with degrees of freedom equal to the sum of the degrees of freedom of the separate components*. This property will be established in the sequel. (Theorem IV).

10. *Addition theorems and limiting forms.* Each of the distributions studied in §9 satisfies a reproductive law. These laws can be derived easily by using the uniqueness axiom of §4. For brevity and clarity they will be stated as theorems.

Theorem I. Let $x_1, x_2, \cdots, x_N$ be independent and normally distributed variables with means $=0$ and respective variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_N^2$. Then the variable

$$L=\sum_{j=1}^{N} c_j x_j$$

is also normally distributed with mean $=0$ and variance

$$\sigma_L^2=\sum_{j=1}^{N} c_j^2\sigma_j^2.$$

Proof. From (29a) and (31) we have

$$K(w,L)=\frac{w^2}{2}\sum_{j=1}^{N} c_j^2\sigma_j^2=\frac{w^2\sigma_L^2}{2}.$$

Here, as in (a) of §9,

$$\kappa_{2:L}=\sigma_L^2 \quad \text{and} \quad \kappa_{r:L}=0 \text{ for } r=1 \text{ and } r>2.$$

Example 11. Let x_j, $(j=1, 2, \cdots, N)$, be independent and normally distributed variates having the same variance σ^2. Let L be a linear function of the x_j's such that $\sigma_L^2=\sigma^2$. Determine L and the form of its distribution.

Solution.

$$L=\frac{1}{\sqrt{N}}\sum_{j=1}^{N} x_j \text{ is normally distributed.}$$

Theorem II. If the independent variables $x_1, x_2, \cdots, x_N$ have binomial distributions with the common parameter p and with $n_1+1, n_2+1, \cdots, n_N+1$ values respectively, then the variable

$$X=\sum_{j=1}^{N} x_j$$

has also a binomial distribution with parameter p and $n'+1$ values where

$$n'=\sum_{j=1}^{N} n_j.$$

Proof. From (33) and (29a) with $c_j=1$, $(j=1, 2, \cdots, N)$, and L replaced by X, we have

$$K(w,X)=n' \log(q+pe^w)$$

and this is the cumulant characteristic function of a binomial distribution with parameter p and $n'+1$ values.

Theorem III. If the independent variables $x_1, x_2, \cdots, x_N$ have Poisson distributions with respective parameters $m_1, m_2, \cdots, m_N$, then the variable

$$X=\sum_{j=1}^{N} x_j$$

has also a Poisson distribution with parameter

$$m'=\sum_{j=1}^{N} m_j.$$

Proof. It follows from (29a) and (37) that

$$\begin{aligned} K(w,X) &= \sum_{j=1}^{N} m_j(e^w-1) \\ &= m'(e^w-1) \\ &= m'(w+w^2/2!+w^3/3!+\cdots). \end{aligned}$$

Consequently, the rth cumulant of the X distribution is

$$\kappa_{X:r}=m'.$$

Theorem IV. If the independent variables $x_1, x_2, \cdots, x_N$ have Pearson type III distributions with respective parameters (b_1, h), $(b_2, h), \cdots, (b_N, h)$, then the variable

$$X=\sum_{j=1}^{N} x_j$$

has a Pearson type III distribution with parameters (b', h) where

$$b' = \sum_{j=1}^{N} b_j.$$

The proof follows from (29a), (40), and the uniqueness axiom.

Theorem V. The limiting form of a Poisson distribution as $m \to \infty$ is normal.

Proof. We first change to standard units by the transformation $t = (x-m)m^{-\frac{1}{2}}$. Applying (11) with $c = m$ and (10) with $c = m^{-\frac{1}{2}}$ to (35) we have

$$M(w,t) = e^{-wm^{\frac{1}{2}}} \cdot e^{m(e^{wm^{-\frac{1}{2}}}-1)}.$$

Then $$K(w,t) = -m^{\frac{1}{2}}w + m(e^{wm^{-\frac{1}{2}}} - 1)$$

$$= \frac{w^2}{2!} + \frac{1}{m^{1/2}} \frac{w^3}{3!} + \cdots + \frac{1}{m^{r/2-1}} \frac{w^r}{r!} + \cdots .$$

Hence we have $$\kappa_1 = 0,\ \kappa_2 = 1, \cdots,\ \kappa_r = \frac{1}{m^{r/2-1}} .$$

Since $\lim\limits_{r\to\infty} \kappa_r = 0$ the cumulants of t approach the cumulants of the normal distribution (in standard units), and this fact indicates that the theorem is true.

Theorem VI. The sum of N variates each of which has a Poisson distribution with parameter m approaches a normal distribution as $N \to \infty$.

This is a consequence of Theorems III and V.

11. *Sampling theory.* A sample of N observations is defined to be N independent determinations of a variate x whose distribution $f(x)$ is variously called the population, universe, or parent distribution. Before the values of the observations have been determined, we may think of them as N independent variates $x_1, x_2, \cdots, x_N$, each having the distribution f. Hence, from the definition of indepencence (§2), their joint distribution is given by

$$f(x_1)f(x_2)\cdots f(x_N)$$

and this is said to represent the distribution of samples.

Let $\bar{x}$ and s^2 denote, respectively, the mean and variance of a sample. These are given by the formulas*

$$N\,x=\sum_{j=1}^{N} x_j, \quad N\,s^2=\sum_{j=1}^{N}(x_j-\bar{x})^2.$$

A *statistic* is defined to be any function of the observations. In sampling theory, each of the statistics $\bar{x}$ and s^2 is itself a variate.

By letting $c_j=1/N$, $(j=1, 2, \cdots, N)$, in (30) we obtain the following relation between the cumulants of the distribution of sample means and the cumulants of an arbitrary parent distribution of unknown form:

$$\kappa_{r:\bar{x}}=N^{1-r}\kappa_{r:x}. \tag{44}$$

The moments of the distribution of sample means in terms of the moments of the parent distribution can be obtained, readily for $r\leqq 4$, from (44) and §8. Thus we find

$$\nu_{1:\bar{x}}=\nu_{1:x}$$

$$\mu_{2:\bar{x}}=\frac{\mu_{2:x}}{N}, \qquad \mu_{3:\bar{x}}=\frac{\mu_{3:x}}{N^2}, \tag{45}$$

$$\mu_{4:\bar{x}}=\frac{1}{N}\left\{\frac{\mu_{4:x}-3\mu^2_{2:x}}{N^2}\right\}+3\frac{\mu^2_{2:x}}{N^2},$$

whence we obtain

$$\begin{cases} \alpha_{3:\bar{x}}=\dfrac{\alpha_{3:x}}{\sqrt{N}} \\[2ex] \alpha_{4:\bar{x}}=\dfrac{1}{N}\{\alpha_{4:x}-3\}+3. \end{cases} \tag{46}$$

We now consider some conclusions that follow when the parent distribution is specialized. In the remaining theorems of this section it is assumed that the parent distribution is normal with mean zero and variance σ^2.

*The sample variance here defined is not to be confused with an unbiased estimate (from the sample) of the variance in the parent distribution. Such an estimate is given by

$$\frac{1}{N-1}\sum_{j=1}^{N}(x_j-\bar{x})^2$$

which some writers denote by s^2.

Theorem VII. The statistic $\bar{x}$ is normally distributed with mean zero and variance σ^2/N.

This is an immediate consequence of Theorem I with $c_j = 1/N$, $(j = 1, 2, \cdots, N)$. The essential feature of the theorem is that the variable sample mean $\bar{x}$ has a normal distribution; the magnitude of the variance of this distribution is known from the second relation in (45).

Theorem VIII. Let $\bar{x}_1$ and $\bar{x}_2$ be two independent samples with N_1 and N_2 observations, respectively, from the same parent distribution. The statistic $\bar{x}_1 - \bar{x}_2$ is normally distributed with mean zero and variance $\sigma^2(1/N_1 + 1/N_2)$.

The proof follows from Theorem I with $N = 2$ and

$$x_1 \text{ replaced by } \bar{x}_1,\ c_1 \text{ by } 1, \quad \sigma_1^2 \text{ by } \sigma^2/N_1,$$

$$x_2 \text{ replaced by } \bar{x}_2,\ c_2 \text{ by } -1, \quad \sigma_2^2 \text{ by } \sigma^2/N_2.$$

Corollary. The statistic

$$(\bar{x}_1 - \bar{x}_2)\left\{\frac{N_1N_2}{N_1+N_2}\right\}^{1/2}$$

is normally distributed with mean zero and variance σ^2.

Theorem IX. The statistic $\bar{x}^2$ has a Pearson type III distribution with $b = 1/2$, $h = N/(2\sigma^2)$.

Proof. From Theorem VIII the distribution of $\bar{x}$ is

$$df = C_1 e^{-N\bar{x}^2/2\sigma^2} d\bar{x}.$$

By the substitution $u = \bar{x}^2$, this is transformed into

$$df = C_2 e^{-Nu/2\sigma^2} u^{-\frac{1}{2}} du$$

which is a type III distribution with $b = 1/2$, $h = N/(2\sigma^2)$. Its characteristic function is, from (39),

$$M(w, \bar{x}^2) = (1 - 2w\sigma^2/N)^{-\frac{1}{2}}.$$

Theorem X. The variable x^2 has a Pearson type III distribution with $b = 1/2$, $h = 1/(2\sigma^2)$.

The proof, which involves the same change of variables used in the proof of Theorem IX, is left to the reader. The characteristic function of x^2 is

$$M(w, x^2) = (1 - 2w\sigma^2)^{-\frac{1}{2}}.$$

Let $NU^2 = \sum_{j=1}^{N} x_j^2$. Then the characteristic function of U^2 is

$$M(w,U^2) = (1-2w\sigma^2/N)^{-N/2}.$$

Also, $U^2 = s^2 + \bar{x}^2$ is an identity.* It is known[16] that the statistics s^2 and $\bar{x}^2$ in samples from a normal parent distribution are *independent* variates. Using this fact, the distribution of s^2 can be obtained at once.

Theorem XI. The sample variance s^2 has a Pearson type III distribution with $b=(N-1)/2$, $h=N/2$.

Proof. Since s^2 and $\bar{x}$ are independent we have, from the identity above and (14),

$$M(w,U^2) = M(w,s^2)\, M(w,\bar{x}^2).$$

This is, $$(1-2w\sigma^2/N)^{-N/2} = M(w,s^2)(1-2w\sigma^2/N)^{-1/2}$$

whence we obtain

$$M(w,s^2) = (1-2w\sigma^2/N)^{-(N-1)/2}.$$

The uniqueness axiom then establishes the theorem.

It follows from Theorem XI and §9 that the distribution of s^2 can also be obtained from the χ^2 distribution. It is easy to show that the transformation

$$\chi^2 = \frac{Ns^2}{\sigma^2}, \quad d(\chi^2) = \frac{N}{\sigma^2}\, d(s^2), \quad n = N-1,$$

carries (43) into $$C_3 e^{-Ns^2/2\sigma^2}(s^2)^{(N-3)/2} d(s^2)$$

where $$C_3 = \left(\frac{N}{2\sigma^2}\right)^{(N-1)/2} \Big/ \Gamma\left(\frac{N-1}{2}\right).$$

From Theorems VII and XI the function for the joint distribution of the variables $\bar{x}$ and s^2 is

$$dF = C_1 e^{-N\bar{x}^2/2\sigma^2} dx \times C_3 (s^2)^{(N-3)/2} e^{-Ns^2/2\sigma^2} d(s^2).$$

Then the joint distribution function of $\bar{x}$ and s is

(47) $$dF = k e^{-P} s^{N-2} d\bar{x}\, ds, \quad P = N(\bar{x}^2+s^2)/2\sigma^2, \quad k = 2C_1C_3,$$

for $s>0$, being identically zero for $s<0$, since s is the positive square root of the expression defining s^2.

*NU^2 here corresponds to the V^2 in the author's book,[17b] Chapter VII, §5.

12. *The t-distribution of Student and Fisher.* From (47) can be derived the function which describes the distribution of

$$t = \bar{x} n^{\frac{1}{2}} / s$$

where $n = N-1$, and N is the size of samples of which $\bar{x}$ is the variable mean and s is the variable standard deviation. For a fixed value of s, we have $d\bar{x} = sn^{-\frac{1}{2}} dt$. Substitution of these expressions into (47) yields.

$$dF = Cn^{-\frac{1}{2}} e^{-P} s^{N-1} ds\, dt, \quad P = \frac{Ns^2}{2\sigma^2}\left(1 + \frac{t^2}{N-1}\right)$$

for the joint distribution of s and t. Now let

$$u = s\left(1 + \frac{t^2}{N-1}\right)^{\frac{1}{2}}, \qquad ds = \left(1 + \frac{t^2}{N-1}\right)^{-\frac{1}{2}} du.$$

Upon making these substitutions in the expression for dF and integrating on u from 0 to ∞, we obtain

$$Cn^{-\frac{1}{2}}\left\{\int_0^\infty u^{N-1} e^{-(Nu^2/2\sigma^2)} du\right\}\left(1 + \frac{t^2}{N-1}\right)^{-N/2} dt.$$

This simplifies into

$$(48) \qquad C_n\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} dt$$

where
$$1/C_n = \frac{n^{\frac{1}{2}}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.$$

The above function (48) is called the t-distribution of Student and Fisher, or sometimes merely that of Student. This and other distributions used in sampling theory are discussed in recent books.[17]

13. *The central limit theorem and some applications.* A theorem in §10 states that the normal curve is the limiting form of the Poisson distribution. A similar theorem could have been proved there for the binomial distribution. It has been deferred to this section because it can be stated more usefully as the DeMoivre-Laplace theorem which, in turn, is a special case of a famous theorem in the theory of probability. This is the Central Limit Theorem, "the discovery of which by Laplace may be considered as the crowning achievement of his

persistent efforts extending over a period of twenty years...". A proof of Laplace's theorem, in general form, would require somewhat more space devoted to formal analysis than seems desirable in the present paper.* Here, only a restricted form of the theorem will be required and this can be proved easily if we assume that *the limiting form of a characteristic function determines the limiting form of its distribution.*† It will be understood that this assumption will be accepted as a lemma. The special case of the theorem which we shall find useful has been stated by Curtiss[18] substantially as follows:

Theorem XII. Let the independent variates $T_1, T_2, \cdots, T_N$ have identical distributions with means zero and unit standard deviations. Let the characteristic function of T_j exist for w in some neighborhood of $w=0$. Let

$$L=\frac{1}{\sqrt{N}}\sum_{j=1}^{N} T_j.$$

Then

$$\lim_{N\to\infty} P(h_1<L<h_2)\ddagger=\frac{1}{\sqrt{2\pi}}\int_{h_1}^{h_2} e^{-x^2/2}dx \tag{49}$$

uniformly for all h_1 and h_2.

Proof. This will also follow the two demonstrations given by Curtiss.

I. The first uses the Taylor series with remainder for the cumulant characteristic function of T_j which we shall denote by $K(w)$. By (29a) with $c_j=1/\sqrt{N}$ and x_j replaced by T_j, we have

$$\begin{aligned} K(w,L)&=N\,K\left(\frac{w}{\sqrt{N}}\right)\\ &=N\left\{K(0)\frac{w}{\sqrt{N}}K^{(1)}(0)+\frac{1}{2!}\left(\frac{w}{\sqrt{N}}\right)^2K^{(2)}(0)\right.\\ &\qquad\left.+\frac{1}{3!}\left(\frac{w}{\sqrt{N}}\right)^3K^{(3)}(\xi)\right\}, \qquad 0<\xi<\frac{w}{\sqrt{N}}, \end{aligned}$$

where $K^{(r)}(0)$ is the rth derivative of $K(w)$ at the origin. Now $K(0)=0$,

*A proof will be found in Uspensky's book.[8]

†For a more precise statement and proof see [5, pp. 284-289].

‡$P(h_1<L<h_2)$ denotes the probability that $h_1<L<h_2$.

and by hypothesis, $K^{(1)}(0)=\kappa_1=\nu_1=0$, $K^{(2)}(0)=\kappa_2=\sigma^2=1$. Substituting these values we find at once that

$$\lim_{N\to\infty} K(w,L)=\tfrac{1}{2}w^2$$

or

$$\lim_{N\to\infty} M(w,L)=e^{w^2/2},$$

which is the characteristic function of a normal distribution with mean zero and unit standard deviation. By extending our uniqueness axiom to include the lemma stated above, the limiting form of the distribution of L is then normal and this is the conclusion stated more precisely in (49).

II. The second proof is shorter but perhaps less rigorous. From (30) we have

$$\kappa_{r:L}=\frac{1}{N^{r/2-1}}\,\kappa_{r:T},$$

whence we obtain the following results:

$$\kappa_{1:L}=0,\quad \kappa_{2:L}=\kappa_{2:T}=1,$$

$$\lim_{N\to\infty}\kappa_{r:L}=\lim_{N\to\infty}\frac{1}{N^{r/2-1}}\,\kappa_{r:T}=0,\quad r=3,4,\cdots.$$

Thus the cumulants of L approach the values of the cumulants of the normal distribution. This fact can be considered "strong circumstantial evidence" for the truth of (49).

We shall now consider two applications of (49), the first of which is the *DeMoivre-Laplace theorem.*

Theorem XIII. Let the variable x have a binomial distribution as in §9b. Then

$$\lim_{N\to\infty} P\left[h_1<\frac{x-np}{\sqrt{npq}}<h_2\right]=\frac{1}{\sqrt{2\pi}}\int_{h_1}^{h_2}e^{-x^2/2}dx$$

uniformly in h_1 and h_2.

Proof. By (17) we can write

$$\frac{x-np}{\sqrt{npq}}=\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{z_j-p}{\sqrt{pq}}$$

and it is obvious that the variables $(z_j-p)/\sqrt{pq}$ satisfy the conditions imposed on the T_j's in Theorem XII.

*Corollary.** The characteristic function of $t=(x-np)/\sqrt{npq}$ is

$$M(w,t)=\{pe^{qw/\sqrt{npq}}+qe^{-pw/\sqrt{npq}}\}^n$$

and the limiting form of this as $n\to\infty$ is $e^{w^2/2}$.

Theorem XIV. Let $\bar{x}$ be the mean of a sample of N observations from an arbitrary universe which possesses a characteristic function. The distribution of $\bar{x}$ tends to normal as N becomes large.

Proof. From (45) we know that $\nu_{1:\bar{x}}=\nu_1$, $\sigma_{\bar{x}}=\sigma/\sqrt{N}$ where ν_1 and σ denote the mean and standard deviation of the parent distribution. We write the identity

$$\frac{(\bar{x}-\nu_1)\sqrt{N}}{\sigma}=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\frac{(x_j-\nu_1)}{\sigma}$$

and verify readily that the variables $(x_j-\nu_1)/\sigma$ satisfy the conditions imposed on the T_j's in Theorem XII. Hence

$$\lim_{N\to\infty} P\left(h_1<\frac{\bar{x}-\nu_1}{\sigma}\sqrt{N}<h_2\right)=\frac{1}{\sqrt{2\pi}}\int_{h_1}^{h_2} e^{-x^2/2}\,dx$$

uniformly in h_1 and h_2; that is, "the mean of a large sample is approximately normally distributed no matter what the parent distribution may be."

14. *Joint distribution.* The characteristic function of the joint probability distribution $f(x_1, x_2,\cdots, x_N)$ is defined by

$$M(w_1, w_2,\cdots, w_N)=E(e^{w_1x_1+w_2x_2+\cdots+w_Nx_N})$$

and this equals

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{\sum_{j=1}^{N} w_jx_j} f(x_1, x_2,\cdots, x_N)dx_1dx_2\cdots dx_N,$$

or

$$\sum_{x_1}\cdots\sum_{x_N} e^{\sum_{j=1}^{N} w_jx_j} f(x_1, x_2,\cdots, x_N),$$

according as the variables are continuous or discrete. Two examples will now be given, the first of which illustrates the discrete case.

In deriving the familiar expression

$$\rho_{jk}\,\sigma_j\,\sigma_k=np_jp_k$$

*See *American Mathematical Monthly*, Vol. 49, No. 9, p. 614.

for the co-variance between sampling deviations in class frequencies, it is sometimes assumed that the deviation of an observed class frequency from its mean value is compensated, in the same sample, by deviations of the opposite sign in all the other class frequencies distributed in proportion to the mean values. A derivation based upon this assumption is objectionable because the assumption is really superfluous. The following derivation of the desired expression is free from the objectionable assumption.

Suppose we have m classes with associated probabilities $p_1, p_2, \cdots, p_m$ such that

$$\sum_{j=1}^{m} p_j = 1.$$

In n repeated trials, during which the p_j's remain constant, let $f_1, f_2, \cdots, f_m$ denote respectively the number of occurrences in the classes. Thus the f_j's are the variables but there are only $m-1$ variables since $f_m = n - f_1 - f_2 - \cdots - f_{m-1}$. The characteristic function of this distribution of m -1 variables is

$$M(w_1, w_2, \cdots, w_{m-1}) = E\{e^{w_1 f_1 + w_2 f_2 + \cdots + w_{m-1} f_{m-1}}\}$$

$$= \sum_{f_1} \cdots \sum_{f_{m-1}} \frac{n!\,e^{w_1 f_1 + \cdots + w_{m-1} f_{m-1}}}{f_1! \cdots f_{m-1}!(n - f_1 - \cdots - f_{m-1})!}\, p_1^{f_1} \cdots p_{m-1}^{f_{m-1}} (1 - p_1 - \cdots - p_{m-1})^{n - f_1 - \cdots - f_{m-1}}$$

since the distribution is a multinomial one. Hence,

$$M(w_1, w_2, \cdots, w_{m-1}) = \{1 - p_1 - \cdots - p_{m-1} + p_1 e^{w_1} + p_2 e^{w_2} + \cdots + p_{m-1} e^{w_{m-1}}\}^n.$$

By differentiating this function with respect to the appropriate variables we can find the moments about the origin of any one of the variables or the product moments. For example,

$$\left.\frac{\partial M}{\partial w_j}\right|_{w_1 = \cdots = w_{m-1} = 0} = E(f_j) = np_j,\ j = 1, 2, \cdots, m-1.$$

Hence, $E(f_m) = np_m$. Also,

$$\left.\frac{\partial^2 M}{\partial w_j^2}\right|_{w_1 = \cdots = w_{m-1} = 0} = E(f_j^2) = n^2 p_j^2 + \sigma_j^2$$

so that,
$$\sigma_j^2 = np_j(1 - p_j).$$

The co-variance of f_j and f_k is

$$\sigma_{jk} = E(f_j f_k) - E(f_j)\, E(f_k)$$

$$= \left[\frac{\partial^2 M}{\partial w_j \partial w_k} - \frac{\partial M}{\partial w_j} \cdot \frac{\partial M}{\partial w_k} \right]_{w's=0}$$

$$= n(n-1)p_j p_k - np_j np_k,$$

whence, using Proposition 3 (§2), we obtain the desired result

(50) $$\sigma_j \, \sigma_k \, \rho_{jk} = -np_j p_k.$$

We conclude with a reference to an example of the continuous case. Let $f(u,v)$ be the joint distribution function of u and v subject to the condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u,v)\,du\,dv = 1.$$

The characteristic function of $f(u,v)$ is

$$M(w_1,w_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{w_1 u + w_2 v} f(u,v)\,du\,dv,$$

where $w_1 = iy_1$ and $w_2 = iy_2$. A proof of the following theorem, utilizing $M(w_1,w_2)$, is available in the literature[19]:

Theorem XV. If u and v are affected by n equally likely causes of which k are common to both, then the coefficient of correlation between u and v is equal to k/n.

Perhaps it should be remarked that both the above theorem and relation (50) can be established without the use of characteristic functions. A neat derivation of (50), using elementary methods yet avoiding the objectionable assumption, has been given by Hotelling[20, §6].

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The Teachers' Department

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Values of the Trigonometric Ratios of $\pi/5$ and $\pi/10$

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At the time the author obtained the exact values of the trigonometric ratios of $\pi/8$ and $\pi/12$ from a figure,* he failed to observe that the ratios of $\pi/5$ and $\pi/10$ can be found in like manner. A starting point for the latter is

$$\sin \pi/10 = \cos 2\pi/5 = \frac{\sqrt{5}-1}{4}.$$

This value can be found either directly from figure 5 of page 14, *Introduction to the Theory of Equations* by L. Weisner, following the suggestion of problem 15 on that page, or by solving a certain reciprocal equation as suggested in problem 13 of page 65, *Introduction to the Theory of Equations* by N. B. Conkwright.

In the accompanying figure, if we let $AC = 1$, the other lengths can easily be shown to have the following values:

$$AD = \sqrt{5}+1$$

$$CD = \sqrt{5+2\sqrt{5}}$$

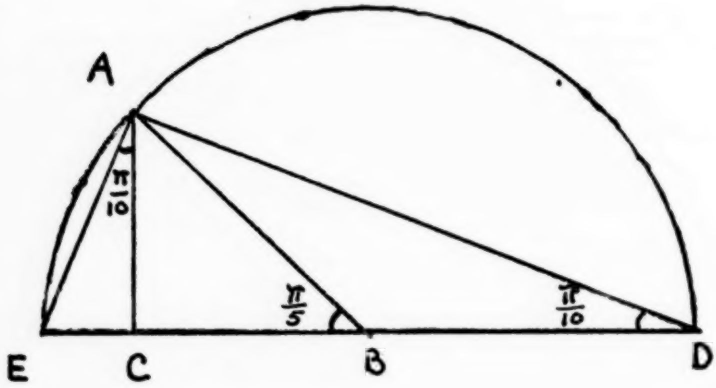
$$CB = \sqrt{25+10\sqrt{5}}/5$$

$$EC = \sqrt{25-10\sqrt{5}}/5$$

$$AB = \sqrt{50+10\sqrt{5}}/5$$

$$AE = \sqrt{50-10\sqrt{5}}/5.$$

**American Mathematical Monthly*, Vol. 49, pp. 324,5.



From triangle ACE we have tan $\pi/10 = EC$ and sec $\pi/10 = AE$, from triangle ACD we have cot $\pi/10 =$ CD and cosec $\pi/10 = AD$, and from triangle ACB we have cot $\pi/5 = CB$ and cosec $\pi/5 = AB$. The other values have to be rationalized. The functions of $\pi/10$ are, of course, the co-functions of $2\pi/5$. For reasons of symmetry we give the latter.

$$\sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4}, \qquad \sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}, \qquad \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$$

$$\tan\frac{\pi}{5} = \sqrt{5-2\sqrt{5}}, \qquad \tan\frac{2\pi}{5} = \sqrt{5+2\sqrt{5}}$$

$$\cot\frac{\pi}{5} = \frac{\sqrt{25+10\sqrt{5}}}{5}, \qquad \cot\frac{2\pi}{5} = \frac{\sqrt{25-10\sqrt{5}}}{5}$$

$$\sec\frac{\pi}{5} = \sqrt{5}-1, \qquad \sec\frac{2\pi}{5} = \sqrt{5}+1$$

$$\csc\frac{\pi}{5} = \frac{\sqrt{50+10\sqrt{5}}}{5}, \qquad \csc\frac{2\pi}{5} = \frac{\sqrt{50-10\sqrt{5}}}{5}.$$

On Various Methods of Solving Cubic Equations

By H. E. Fettis
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1. *Some General Remarks.* The student of advanced algebra will no doubt be familiar with Cardan's method of solving the general cubic equation, as well as with the numerical difficulties involved therein, due to the necessity of first removing the x^2 term before solving. The primary aim of this article is to show how a solution may be effected with no reference to the reduced equation, and with a corresponding reduction in the complications involved. The method is not, to the author's knowledge, in general use, although the theory behind it is well known.

2. *Reduction of the General Cubic to the Ratio of Cubes.* The method used here is essentially that given by Dr. Morley:* Let the given equation be

$$ax^3+3bx^2+3cx+d=0 \tag{1}$$

it being required to find two numbers h_1 and h_2 such that

$$(x-h_1)^3=K(x-h_2)^3 \tag{2}$$

where, in the process, K must also be found.

Expanding (2) and collecting like powers of x,

$$x^3(1-K)-3x^2(h_1-Kh_2)+3x(h_1^2-Kh_2^2)^2-(h_1^3-Kh_2^3)=0. \tag{3}$$

Comparing with (1), we must have

$$a=1-K, \quad -b=h_1-Kh_2, \quad c=h_1^2-Kh_2^2, \quad -d=h_1^3-Kh_2^3$$

Forming products, and subtracting those with like powers of h:

$$ac-b^2=-Kh_1^2+2Kh_1h_2-Kh_2^2=-K(h_1-h_2)^2$$

$$ad-bc=Kh_2^3-Kh_1^2h_2-Kh_1h_2^2+Kh_1^3=K(h_1-h_2)^2(h_1+h_2)$$

$$bd-c^2=-Kh_1^3h_2+2Kh_1^2h_2^2-Kh_1h_2^3=-K(h_1-h_2)^2(h_1h_2)$$

***Inversive Geometry*, Article 42, Ginn & Co. 1933.

This gives

$$h_1+h_2=-\frac{ad-bc}{ac-b^2}$$

$$h_1h_2=\frac{bd-c^2}{ac-b^2}$$

from which we see that h_1 and h_2 are roots of the quadratic

$$(ac-b^2)h^2+(ad-bc)h+(bd-c^2)=0 \tag{4}$$

Thus we may assume values to be known for h_1 and h_2; it remains to find K, which may now be expressed in terms of h. First considerations might lead us to suppose that it could be found from the original identities, but since there is actually an extraneous constant involved in the numbers a, b, c, and d, it follows that K can only be uniquely determined by taking the quotient of two of these identities. The simplest being that of the first two, we have

$$-\frac{b}{a}=\frac{h_1-Kh_2}{1-K}.$$

Let $-\dfrac{b}{a}=g$, and solve the above for K, obtaining

$$K=\frac{g-h_1}{g-h_2}.$$

The solution of the given equation is therefore

$$\frac{x-h_1}{x-h_2}=\sqrt[3]{\frac{g-h_1}{g-h_2}}.$$

3. *Modifications.* A little consideration discloses the fact that the above relation could only be used to advantage in the exceptional instance when K is a rational number with an exact cube root. We therefore seek a more usuable result, expressing x explicitly. To this end, let

$$g-h_1=m^3, \quad g-h_2=n^3.$$

Then $$x-h_1=(x-g)+m^3, \quad x-h_2=(x-g)+n^3$$

and we find from the formula that

$$\frac{(x-g)+m^3}{(x-g)+n^3}=\frac{m}{n}$$

$$n(x-g)+m^2n=m(x-g)+mn^2 \qquad (x-g)(m-n)=mn(m^2-n^2),$$

$$x-g=m^2n+mn^2.$$

Substituting for m and n their original values, we have the desired result

$$x_1=g+\sqrt[3]{(g-h_1)(g-h_2)^2}+\sqrt[3]{(g-h_1)^2(g-h_2)}$$

Using the complex cube roots of K, we find the other values of x to be

$$x_2=g+w\sqrt[3]{(g-h_1)(g-h_2)^2}+w^2\sqrt[3]{(g-h_1)^2(g-h_2)}$$

$$x_3=g+w^2\sqrt[3]{(g-h_1)(g-h_2)^2}+w\sqrt[3]{(g-h_1)^2(g-h_2)}$$

which are the expressions best adapted to the numerical computation of the roots. Actually, the formulæ may be modified somewhat, but they are more easily retained in this symmetric form.

Equation (4) is the *Hessian* of the given cubic, and can therefore be expressed in determinant form as

$$\begin{vmatrix} 1 & a & b \\ -x & b & c \\ x^2 & c & d \end{vmatrix}=0.$$

We may now outline the complete process, taking the given equations

$$ax^3+3bx^2+3cx+d=0$$

(1) Compute h_1 and h_2, the roots of

$$\begin{vmatrix} 1 & a & b \\ -x & b & c \\ x^2 & c & d \end{vmatrix}=0.$$

(These should be kept in radical form, when irrational.)

(2) Taking g as $\dfrac{b}{a}$, find the respective values of

$$g-h_1, \quad g-h_2.$$

(3) Evaluate the product

$$(g-h_1)(g-h_2)$$

(4) Multiply this product in turn by each of the quantities $g-h_1$ and $g-h_2$ to obtain the quantities (A and B) under the radicals. Then

$$x_1=g+\sqrt[3]{A}+\sqrt[3]{B}, \quad x_2=g+w\sqrt[3]{A}+w^2\sqrt[3]{B}, \quad x_3=g+w^2\sqrt[3]{A}+w\sqrt[3]{B}.$$

(For numerical examples, see end of article.)

We note that the product of $g-h_1$ and $g-h_2$ will always be rational. It can, in fact, be found without knowing the value of either of the two quantities themselves, but it is usually better to find it by actual multiplication, when feasible, using the relation which we are about to develop as a check.

By actual multiplication

$$(g-h_1)(g-h_2)=g^2-g(h_1+h_2)+h_1h_2.$$

But
$$g=-\frac{b}{a},\quad h_1+h_2=\frac{ad-bc}{ac-b^2},\quad h_1h_2=\frac{bd-c^2}{ac-b^2}.$$

Substituting these values, we find, after some reduction, that

$$(g-h_1)(g-h_2)=-\frac{ac-b^2}{a^2}.$$

In words, the product equals the coefficient of h^2, with sign changed, over a^2, a relationship which gives us both an easy method of obtaining this product, when actual multiplication would be too difficult, and a convenient check on the correctness of the work to this point.

4. *Derivation of Cardan's Form.* For the reduced cubic, we may take the form to be

$$x^3+3px+2q=0.$$

The Hessian of this is

$$\begin{vmatrix} 1 & 1 & 0 \\ -h & 0 & p \\ h^2 & p & 2q \end{vmatrix}=0$$

or
$$ph^2+2qh-p^2=0$$

$$h_1=\frac{-q+\sqrt{q^2+p^3}}{p},\qquad h_2=\frac{-q-\sqrt{q^2+p^3}}{p}.$$

Furthermore, since $g=0$, we have

$$x_1=\sqrt[3]{-h_1h_2^2}+\sqrt[3]{-h_1^2h_2},\ \text{etc.}$$

and since $h_1h_2=-p$, this becomes

$$x_1=\sqrt[3]{-q+\sqrt{q^2+p^3}}+\sqrt[3]{-q-\sqrt{q^2+p^3}},\ \text{etc.}$$

the well known formula of Cardan.

5. *Nature of the Roots—The Discriminant.* For the quadratic equation

$$ax^2+bx+c=0$$

it is sometimes customary to speak of the discriminant as the quantity

$$b^2-4ac$$

which is positive, zero, or negative according as the roots are real and distinct, equal, or imaginary. For the cubic equation we may define the discriminant to be the symmetric function of the roots

$$\Delta=(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2$$

and a little reflection indicates that this will be positive if all roots are real, zero if two are equal, and negative if two are complex. This would suggest that a similar relation existed between the roots and the discriminant of the quadratic; and it is easily shown that for the quadratic above.,

$$\Delta=(x_1-x_2)^2=\frac{b^2-4ac}{4a^2}.$$

The two are seen to differ only by the factor $1/4a^2$, and since the presence of this factor can affect only the numerical value, and not the sign, we conclude that both views are equally satisfactory. We will, however, use the latter definition, as it is more consistent with theory.

To compute Δ for the cubic, let us return to the abridged notation employed in Art. 3. We have

$$x_1=g+m^2n+mn^2,\quad x_2=g+wm^2n+w^2mn^2,\quad x_3=g+w^2m^2n+wmn^2,*$$

$$x_1-x_2=m^2n(1-w)+mn^2(1-w^2)=(1-w)(m^2n-w^2mn^2)$$

$$x_2-x_3=m^2n(w-w^2)+mn^2(w^2-w)=w(1-w)(m^2n-mn^2)$$

$$x_1-x_3=-m^2n(1-w^2)-mn^2(1-w)=-w^2(1-w)(m^2n-wmn^2)$$

$$(x_1-x_2)(x_2-x_3)(x_1-x_3)=-3\sqrt{-3}(m^3n^3)(m^3-n^3).$$

Substituting $m^3=g-h_1$ and $n^3=g-h_2$, and squaring

$$\Delta=-27(g-h_1)^2(g-h_2)^2(h_1-h_2)^2.$$

Now the quantity $(h_1-h_2)^2$ is precisely the discriminant of the Hessian, and since $(g-h_1)(g-h_2)$ is real, the only material difference between the former and Δ will be in the sign. In other words, the equation

*The writer uses $w=\omega$, the usual symbol for an imaginary cube root of 1.

will have real, (equal) or complex according as the Hessian has complex, (equal) or real roots.

6. *The Case of Three Real Roots.* It appears from Art. 5 that this case requires the cube roots of complex quantities, the finding of which is impossible (except in rare cases) if we are limited to purely algebraic operations. This form is then algebraically irreducible, but the solution may be effected by the aid of trigonometry.

Demoivre's Theorem gives

$$\sqrt[3]{\cos\Theta+i\sin\Theta}=\cos\Theta/3+i\sin\Theta/3.$$

To apply this to the general complex number

$$a+ib$$

we first convert it into the form $r(\cos\Theta+i\sin\Theta)$ by means of the relations

$$r^2=a^2+b^2;\quad \Theta=\text{arc tan } b/a=\text{arc cos } a/r.$$

Then $$\sqrt[3]{a+ib}=\sqrt[3]{r}(\cos\Theta/3+i\sin\Theta/3).$$

Applying this to the case at hand, let

$$g-h_1=r(\cos\Theta+i\sin\Theta).$$

Then $$g-h_2=r(\cos\Theta-i\sin\Theta)$$

$$(g-h_1)(g-h_2)=r^2$$

$$\begin{aligned}x_1&=g+\sqrt[3]{r^3(\cos\Theta+i\sin\Theta)}+\sqrt[3]{r^3(\cos\Theta-i\sin\Theta)}\\&=g+r(\cos\Theta/3+i\sin\Theta/3)+r(\cos\Theta/3-i\sin\Theta/3)\\&=g+2r\cos\Theta/3.\end{aligned}$$

Since $w=\cos 120°+i\sin 120°$, $w^2=\cos 240°+i\sin 240°$,

$$x_2=g+2r\cos(\Theta/3+120°)$$

$$x_3=g+2r\cos(\Theta/3+240°).$$

7. *Other Methods Based on Trigonometry.* When a cubic equation lacks the x^2-term, it may always be solved by a direct trigonometric process, with no actual reference to the algebraic method. Let the equation be

$$x^3+3px+2q=0.$$

Substituting $r \cos \Theta$ for x, this may be written in the form

$$\cos^3\Theta + \frac{3p}{r^2} \cos \Theta + \frac{2q}{r^3} = 0.$$

Comparing this with the familiar identity

$$\cos^3\Theta + \tfrac{3}{4} \cos \Theta = \tfrac{1}{4} \cos 3\Theta$$

we see that the two will be identical if we take

$$-\frac{3p}{r^2} = \tfrac{3}{4}; \qquad -\frac{2q}{r^3} = \frac{\cos 3\Theta}{4},$$

that is, if we take

$$r = 2\sqrt{-p}; \quad \cos 3\Theta = \sqrt{-\frac{q^2}{p^3}}.$$

It is evident that for a solution to be possible by this method, p must be negative and $\cos 3\Theta < 1$, and since these are precisely the conditions which make $\Delta > 0$, it follows that the cubic with three real roots can always be solved by this method. The angle Θ, as determined by this process is, in fact, the same as the one found by the method of Art. 6.

The method would supposedly fail to give a solution when $\Delta < 0$, since we would then either have p positive, or $\cos 3\Theta > 1$. In either case a solution may be obtained by means of hyperbolic functions; it is necessary, however, to consider separately the two instances.

Since $$\sinh 3\Theta = 4 \sinh^3\Theta + 3 \sinh \Theta$$

and since there are no restrictions of the value of a hyperbclic sine, it follows that the first of these two cases may be handled by using $\sinh \Theta$ in place of $\cos \Theta$. The difficulty of having $\cos 3\Theta > 1$ is taken care of by using the substitution $r \cosh \Theta$, since the value of this function is always greater than 1, and the relation connecting $\cos 3\Theta$ and $\cosh \Theta$ is identical to the one involving the circular functions.

The above methods are not essentially different from Cardan's method, as may be shown by substituting $r \sinh \Theta$ for x in

$$x^3 + 3px + 2q = 0.$$

Proceeding as before, we find that

$$r = 2\sqrt{p}$$

$$\sinh 3\Theta = \frac{-q}{p\sqrt{p}}$$

and since $$\text{arc sinh}\, y = \log(y+\sqrt{y^2+1})$$

we have $$3\Theta = \log\left(\frac{-q}{p\sqrt{p}} + \frac{\sqrt{q^2+p^3}}{p\sqrt{p}}\right)$$

$$\Theta = \log \sqrt[3]{\frac{-q+\sqrt{q^2+p^3}}{p\sqrt{p}}}$$

$$e^{\Theta} = \frac{\sqrt[3]{-q+\sqrt{q^2+p^3}}}{\sqrt{p}}, \qquad e^{-\Theta} = \frac{\sqrt[3]{-q-\sqrt{q^2+p^3}}}{-\sqrt{p}}$$

$$\sinh\Theta = \tfrac{1}{2}(e^{\Theta}-e^{-\Theta}) = \frac{\sqrt[3]{-q+\sqrt{q^2+p^3}} + \sqrt[3]{-q-\sqrt{q^2+p^3}}}{2\sqrt{p}}$$

$$x = r\sinh\Theta = 2\sqrt{p}\,\sinh\Theta = \sqrt[3]{-q+\sqrt{q^2+p^3}} + \sqrt[3]{-q-\sqrt{q^2+p^3}}.$$

It is easily seen that any other number could have been used in place of e*, and this principle enables us to apply the process, without any direct reference to tables of hyperbolic functions. Thus we avoid the difficulty of attempting to interpolate in such tables, in which the intervals are usually too large to give accurate results. It is necessary to first explain the so-called gudermannian relation, by means of which hyperbolic functions are converted into circular ones.

Since $\cosh\Theta > 1$, there will always exist an angle y, such that

$$\sec y = \cosh\Theta$$

y is called the *gudermannian* of Θ, written

$$y = gd\ \Theta.$$

Then because of the relation

$$\cosh^2\Theta - \sinh^2\Theta = 1$$

it follows that $$\tan y = \sinh\Theta.$$

*That is, we may think of the substitution of

$$\sinh\Theta = \tfrac{1}{2}(e^{\Theta}-e^{-\Theta})$$

to be replaced by a similar relation involving 10, or some other more convenient number than e. In general this would not be permissible, but in this particular instance we are concerned not in the actual value of Θ or $\sinh\Theta$, but in the behavior of the function, which would not be affected by the change of base. It will be noted in the examples that in each case we find the same values for $\sinh 3\Theta$ and $\sinh\Theta$, but that the quantity Θ is different.

Thus when a hyperbolic function is given, the gudermannian is easily found. To find Θ when y is known, we have

$$\cosh\Theta = \sec y, \quad \sinh\Theta = \tan y$$

$$\sinh\Theta + \cosh\Theta = \tan y + \sec y, \quad e^{\Theta} = \tan y + \sec y$$

$$\Theta = \log(\tan y + \sec y) = \log\left(\frac{\sin y + 1}{\cos y}\right) = \log\tan(45° + y/2).$$

Ordinarily, this would have to be a natural logarithm, but, (as explained above), for the purposes at hand, common logs will suffice. Details of the process are worked out in Art. 9.

EXAMPLES

8. *Solutions by means of the Hessian.* Case 1. When h_1 and h_2 are rational.

$$11x^3 + 21x^2 + 33x + 13 = 0.$$

The Hessian equation is

$$\begin{vmatrix} 1 & 11 & 7 \\ -h & 7 & 11 \\ h^2 & 11 & 13 \end{vmatrix} = 0$$

or $$72h^2 + 66h - 30 = 0$$

whose roots are $$h_1 = \tfrac{1}{3}; \quad h_2 = -(5/4).$$

Then since $g = -(7/11)$, we have

$$g - h_1 = -(32/33), \quad g - h_2 = 27/44$$

$$(g - h_1)(g - h_2) = -(72/121) \quad \text{(as it should be by ¶3)}$$

$$x = \frac{-7}{11} + \sqrt[3]{\frac{24\ 32}{11^3}} + \sqrt[3]{\frac{18\ 27}{11^3}}$$

$$x = -.518592.$$

Case II. When h_1 and h_2 are real but irrational.

Ex. $3x^3 - 5x^2 + 2x - 4 = 0$

$$\begin{vmatrix} 1 & 9 & -5 \\ -h & -5 & 2 \\ h^2 & 2 & -12 \end{vmatrix} = 0$$

$$-7h^2 - 98h + 56 = 0$$

$$h=-7\pm\sqrt{57};\quad g=5/9$$

$$g-h=\frac{68\mp 9\sqrt{57}}{9}$$

$$(g-h_1)(g-h_2)=7/81$$

$$x=\frac{5+\sqrt[3]{476+63\sqrt{57}}+\sqrt[3]{476-63\sqrt{57}}}{9}$$

$$=\frac{5+\sqrt[3]{951.6395}+\sqrt[3]{.3605}}{9}$$

$$=\frac{5+9.8360+.7117}{9}$$

$$x=1.7275.$$

Case III. When h_1 and h_2 are complex.

Ex. 1. $x^3-4x^2-x+3=0.$

$$\begin{vmatrix} 1 & 3 & -4 \\ -h & -4 & -1 \\ h^2 & -1 & 9 \end{vmatrix}=0$$

$$-19h^2+23h-37=0$$

$$h=\frac{23\pm\sqrt{-2283}}{38};\quad g=4/3$$

$$g-h=\frac{83\mp 3\sqrt{-2283}}{3x38}$$

$$r^2=(g-h_1)(g-h_2)=19/9$$

$$r=\frac{\sqrt{19}}{3}$$

$$\log r=.16226$$

$$\tan 3\theta=\frac{3\sqrt{2283}}{83}$$

$$\log\tan 3\theta=.237295$$

$$3\theta=59°\ 55'\ 40''$$

$$
\begin{aligned}
\theta &= 19^\circ\ 58'\ 33'' \\
\log\cos\theta &= 9.97305 \\
\log r &= .16226 \\
\log 2 &= \underline{.30103} \\
\log 2r\cos\theta &= .43634 \\
2r\cos\theta &= 2.73113 \\
g &= \underline{1.33333} \\
x_1 &= 4.06446 \\
\theta' = 120+\theta &= 139^\circ\ 58'\ 33'' \\
\log\cos\theta' &= 9.88410 \\
\log r &= .16226 \\
\log 2 &= \frac{.30103}{.34739} \\
2r\cos\theta' &= -2.22530 \quad \text{(since } \cos\theta' \text{ is } -) \\
g &= \underline{1.33333} \\
x_2 &= -.89197 \\
\theta'' = 240+\theta &= 259^\circ\ 58'\ 33'' \\
\log\cos\theta'' &= 9.24071 \\
\log r &= .16226 \\
\log 2 &= \underline{.30103} \\
& \ 9.70400 \\
2r\cos\theta'' &= -.50582 \\
g &= \underline{1.33333} \\
x_3 &= .82751
\end{aligned}
$$

9. *Direct Trigonometric Solutions.* Case I. Three Real Roots

$$7x^3-4x-1=0.$$

Replacing x by $r\cos\theta$, and comparing with the trigonometric identity

$$\frac{4}{7r^2}=\tfrac{3}{4}; \qquad \frac{1}{7r^3}=\frac{\cos 3\theta}{4}.$$

Hence $$r^2=\frac{16}{21}\ ; \qquad \cos 3\Theta=\frac{4}{7r^3}$$

from which we can easily compute

$$\begin{aligned}
\log r &= 9.94095 \\
\log \cos 3\Theta &= 9.93411 \\
3\Theta &= 30^\circ\ 46'\ 9'' \quad \text{or} \quad 390^\circ\ 46'\ 9'' \quad \text{or} \quad 750^\circ\ 46'\ 9'' \\
\Theta &= 10^\circ\ 15'\ 23'' \quad \text{or} \quad 130^\circ\ 15'\ 23'' \quad \text{or} \quad 250^\circ\ 15'\ 23'' \\
\log \cos \Theta &= 9.99300 \ \text{or}\ 9.81038 \ \text{or}\ 9.52867 \\
\log r &= \underline{9.94095}, \quad \underline{9.94095}, \quad \underline{9.94095} \\
\log x &= 9.93395 \ \text{or}\ 9.75133 \ \text{or}\ 9.46962
\end{aligned}$$

$$x_1=.85892, \quad x_2=-.56406, \quad x_3=-.29486$$

(The results may be checked in part by observing that their sum is 0, and that the sum of their logs is the log of 1/7.)

Case II. One Real Root; $\cos 3\Theta>1$

$$x^3-2x-2=0.$$

Proceeding as above, we find

$$\frac{2}{r^2}=3/4; \qquad \frac{2}{r^3}=\frac{\cosh 3\Theta}{4}$$

$$r^2=8/3; \quad \cosh 3\Theta=\frac{8}{r^3}$$

$$\begin{aligned}
\log r &= .21299 \\
\log \cosh 3\Theta &= .264135 \\
\cosh 3\Theta &= 1.8471 \\
3\Theta &= 1.21734 \\
\Theta &= .40578 \\
\cosh \Theta &= 1.08346 \\
\log \cosh \Theta &= .03481 \\
\log r &= \underline{.21299} \\
\log x &= .24780 \\
x &= 1.7693
\end{aligned}$$

In the event that tables of hyperbolic functions are not available, an alternative method based on the gudermannian relation may be used. We outline the process briefly, leaving the reader to fill in the details.

(1) Having found $\log \cosh 3\Theta (= \log \sec gd3\Theta)$ compute from trigonometric tables the value of $gd3\Theta$.

(2) Next find 3Θ by the relation: $3\Theta = \log \tan(45 + \frac{1}{2}gd3\Theta)$, using common logs, throughout.

(3) Find Θ itself, and reverse the process to obtain $gd\Theta$ and $\log \cosh \Theta$. When $\log \cosh \Theta$ is known, the remainder of the solution proceeds as before.

Case III. One Real Root; r^2 Negative.

$$2x^3 + 2x - 1 = 0.$$

Here the procedure is the same as for Case II, except that $\sinh \Theta$ replaces $\cosh \Theta$ throughout. In this case, also, we may either use tables of hyperbolic function, or if preferred, the method based on the gudermannian. In either case the result from five place tables is

$$x = .42386.$$

Case IV. The General Equation. Although the trigonometric methods discussed are essentially solutions for the reduced equation, they may be used to obtain the roots of an equation in the general form. By the theory of symmetric functions of the roots,

$$g = \tfrac{1}{3}(x_1 + x_2 + x_3)$$

from which it may be seen that if an equation is found such that its roots are each less by g than those of the original, the g of the new equation will be zero; that is, the equation will lack the term in x^2. A convenient method of doing this is to divide three times by the quantity $x = g$, as in Horner's method.

Applying this to the equation

$$x^3 - x^2 - x - 5 = 0$$

we would have $g=\frac{1}{3}$, and the work would be done as follows

$$
\begin{array}{rrrr|l}
1 & -1 & -1 & -5 & \frac{1}{3} \\
 & \frac{1}{3} & -\frac{2}{9} & -\frac{11}{27} & \\
\hline
1 & -\frac{2}{3} & -\frac{11}{9} & -\frac{146}{27} & \\
 & \frac{1}{3} & -\frac{1}{9} & & \\
\hline
1 & -\frac{1}{3} & -\frac{12}{9} & & \\
 & \frac{1}{3} & & & \\
\hline
1 & 0 & & &
\end{array}
$$

Thus the reduced equation is

$$y^3-\frac{12}{9}y-\frac{146}{27}=0$$

from which we can compute

$$
\begin{aligned}
\log r &= .12494 \\
\log\cosh 3\theta &= .96023 \\
\text{Hence}\qquad \log\cosh\theta &= .17759 \\
\log r &= \underline{.12494} \\
\log y &= .30253 \\
y &= 2.00691 \\
g &= .33333 \\
x &= 2.34024
\end{aligned}
$$

A direct trigonometric solution may also be used when the equation lacks the x-term, by setting $x=1/y$; the resulting equation will then be in the reduced form. Thus the equation

$$x^3-2x^2-2=0 \quad \text{would become} \quad 2y^3+2y-1=0$$

and since the roots of the latter are the reciprocals of the roots of the original, we find x from the relation

$$\log x=\operatorname{colog} y.$$

Announcements

Most of the changes noted in what follows are already indicated in our editorial roster:

Agreeably to G. Waldo Dunnington, the head of our History Department, A. W. Richeson, of the University of Maryland, accepted our invitation to become a member of this Department. Though he was entering a division of the war service, Editor Dunnington yet wished to retain his official relation to the MAGAZINE. Professor Richeson was glad to take over most of the Department burden for the duration.

Jno. W. Cell, for some time associated with Editor H. A. Simmons, chairman of the Book Review Department, found it necessary to resign his duties with the MAGAZINE to undertake those of the Secretaryship of the S. P. E. E. to which he was recently elected. In his letter of resignation, he writes: "I shall retain a keen interest in the MAGAZINE, and shall continue my subscription."

On the suggestion of Chairman Simmons, P. K. Smith, of Louisiana Polytechnic Institute, was invited to take the place of Jno. W. Cell. Editors Simmons and Smith had been associated together for several years prior to the period of Editor Cell's MAGAZINE service. Professor Smith was glad again to join the N. N. M. editorial group.

Robert C. Yates, who for a number of years had headed our Problem Department, and, who with the fine assistance of E. P. Starke, of Rutgers, had built up an excellent division of the MAGAZINE, has also been compelled to sever his connection with this journal. It was a strong hope with him that his call to service as a faculty member of the U. S. Military Academy would not require his detachment from MAGAZINE work. But, a wholly unanticipated family bereavement resulted in a situation which has made the detachment necessary. His resignation was recently tendered.

Loyal to our journal, Emory P. Starke, with not the slightest hesitation, accepted our urgent invitation to the Chairmanship of the

very important Problem Department in the place of R. C. Yates, resigned.

But E. P. Starke, despite a highly competent scholarship, is, too hard-pressed with extra war duties at Rutgers, to administer without adequate help the Problem division of the MAGAZINE. Therefore, his earnest request for an associate in the Department has been granted. Desiring especially that this associate, if possible, should represent the field of modern geometry, he was enthusiastically agreeable to the suggestion that we invite N. A. Court,of the University of Oklahoma—a mathematician who, today is, easily, a topmost figure among Americans devoting their research and their teaching to modern geometry. His printed name at the head of our Problem Department of this issue of the MAGAZINE is evidence that he has accepted our invitation.

S. T. SANDERS.

Problem Department

Edited by
E. P. STARKE and N. A. COURT*

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscripts be typewritten with double spacing. Send all communications to EMORY P. STARKE, Rutgers University, New Brunswick, N. J.

SOLUTIONS

No. 419. Proposed by *Paul D. Thomas*, student, Oklahoma University.

A variable line meets two fixed skew lines in points P and Q such that angle PRQ is a right angle, where R is the midpoint of the common perpendicular to the skew lines. (a) Find the locus of the line PQ. (b) Find the locus of the midpoint of PQ.

Solution by the *Proposer*.

Choose R as origin, the common perpendicular of the skew lines as Z-axis, and the plane through R parallel to each of the skew lines as XY-plane. For X-axis take the bisector of the angle Θ between the projections of the skew lines upon the XY-plane. Then any point on one of the skew lines is P, (a, ma, c), and any point on the other is Q, $(b, -mb, -c)$, where $2c$ is the length of the common perpendicular of the skew lines, $m=\tan \frac{1}{2}\Theta$, and a, b are parameters.

The condition that angle PRQ be a right angle is $\overline{PR}^2+\overline{RQ}^2=\overline{PQ}^2$, which leads at once to

$$(1) \qquad ab(1-m^2)=c^2.$$

The hypothesis excludes 1 and -1 as values for m since then (1) implies $c=0$, whence the fixed lines intersect. Similarly m is not 0, for then the lines would be parallel. The equations of PQ are clearly

$$(2) \qquad (x-a)/(a-b)=(y-ma)/m(a+b)=(z-c)/2c.$$

*E. P. Starke is responsible for the December Problem Department. N. A. Court becomes officially active with the January issue.—ED

The elimination of a, b from (1) and (2) gives

$$(1-m^2)(m^2x^2-y^2)=m^2(c^2-z^2), \tag{3}$$

the equation of the locus (a), which for all admissible values of m is an hyperboloid of one sheet.

The midpoint of PQ is M, $(\frac{1}{2}(a+b),\ \frac{1}{2}m(a-b),\ 0)$. Thus M lies in the plane $z=0$. Hence equation (3) determines the locus (b) as the hyperbola

$$(1-m^2)(m^2x^2-y^2)=m^2c^2, \quad z=0.$$

No. 430. Proposed by *N. A. Court*, University of Oklahoma.

Four given spheres have a point E in common and intersect three by three in the points A', B', C', D'. The chords EA', EB', EC', ED' common to triads of given spheres meet the respective fourth given sphere in the points A'', B'', C'', D''. Prove that the segments $A'A''$, $B'B''$, $C'C''$, $D'D''$ are twice as long as the altitudes of the tetrahedron formed by the centers of the given spheres.

Solution by the *Proposer*.

The common chord ED' of the three given spheres $(A)=EB'C'D'$, $(B)=EC'D'A'$, $(C)=ED'A'B'$ is perpendicular to the plane ABC of their centers A, B, C; and if P is the trace of ED' in ABC, we have, both in magnitude and in sign, $EP=PD'$.

Again, the two points E, D'' of the line $EPD'D''$ lie on the fourth given sphere $(D)=EA'B'C'$; hence, if Q is the foot of the perpendicular from the center D of (D) upon the line $EPD'D''$, we have, both in magnitude and in sign, $EQ=QD''$.

Adding the two equalities we have

$$PE+EQ=D'P+QD''=(D'Q+QP)+(QD'+D'D'')=QP+D'D'',$$

or

$$2PQ=D'D''.$$

Now PQ is equal (and parallel) to the altitude of the tetrahedron $DABC$ issued from the vertex D, and the other altitudes of this tetrahedron may be treated in a like manner. Hence the proposition.

NOTE: The corresponding proposition in the plane is due to the Belgian mathematician J. Neuberg (1840-1926), *Educational Times, Reprints*, Vol. 54 (1891), p. 102, Q. 10699.

No. 452. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

Through a fixed point P in the plane of a given angle A draw the line that forms with the sides of the angle a triangle ABC such that $b+c-a$ is a minimum.

Solution by *W. B. Clarke*, San Jose, California.

Let a line through P intersect the sides of the given angle A in the points B and C. For definiteness, suppose AB is the nearer side of the angle; i. e. the distance k of P from AB is not greater than its distance from AC. Let the circle with center B and radius BC cut AB at D. Clearly $AD=c-a$, so that $b+c-a$ is represented as the sum of the segments AD and AC. If the line BC is rotated about P so as to shorten AC, then also AD is shortened and $b+c-a$ is diminished. The process fails however when CP becomes parallel to AB. In this limiting situation B has gone infinitely far off, arc CD has become a straight line of length k perpendicular to AB, and

$$b+c-a=k(\operatorname{cosec} A+\cot A)=k \cot \tfrac{1}{2}A.$$

Strictly there is no triangle for which $b+c-a$ is a minimum, but there are triangles such that $b+c-a$ exceeds $k \cot \frac{1}{2}A$ by an arbitrarily small amount. $k \cot \frac{1}{2}A$ is the greatest lower bound of $b+c-a$.

If the rotation mentioned above is continued beyond the position of parallelism to AB, then point C lies on the side of the angle extended beyond the vertex. If this is deemed admissible under the hypothesis, then the triangle ABC can be made indefinitely small so that $b+c-a$ will have zero for its lower limit.

EDITOR'S NOTE. The above solution requires modification if the given angle is not acute. The Proposer gave an ingenious but lengthy proof involving calculus. The following remark of his is helpful in connection with the above. The circle inscribed in triangle ABC is smallest in the limiting situation when CP is parallel to AB, and largest when the circle passes through P with CP tangent at P. As is well known, $b+c-a$ is twice the distance from A to the point of tangency of the circle with AB (or with AC) and has its extreme values corresponding to the two circles just mentioned. The greatest lower limit of $b+c-a$ is $k \cot \frac{1}{2}A$ as is easily seen from a figure. This argument applies equally well when A is not an acute angle.

No. 464. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

Show that for any right triangle ABC the sum of the distances of the orthocenter H from the vertices equals the sum of the diameters of the inscribed and circumscribed circles.

I. Solution by *P. D. Thomas*, Lucedale, Mississippi.

In any triangle (right or oblique) the sum of the distances of the circumcenter from the three sides of the triangle is equal to the circumradius increased by the inradius.* But the distance of a side of a triangle from the circumcenter is equal to half the distance of the opposite vertex from the orthocenter of the triangle.†

Hence in any triangle the sum of the distance to the vertices from the orthocenter is twice the sum of the inradius and circumradius, or the sum of the respective diameters.

II. Solution by the *Proposer*.

If AD is an altitude, angle BHD equals angle C, $BD = c \cos B$, and $BH = BD/\sin C$. Since $\sin C = c/2R$, it follows that $BH = 2R \cos B$, which, with analogous values for AH and CH, gives

$$AH+BH+CH=2R(\cos A+\cos B+\cos C). \tag{1}$$

Into the familiar identity

$$\cos A+\cos B+\cos C=1+4 \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$$

put

$$\sin^2 \tfrac{1}{2}A=(s-b)(s-c)/bc$$

etc., to transform the right member of (1) into

$$2R+8R(s-a)(s-b)(s-c)/abc.$$

This is finally reduced by the relation

$$\sqrt{s(s-a)(s-b)(s-c)}=sr=abc/4R$$

to the desired result

$$AH+BH+CH=2R+2r.$$

Also solved by *Emma E. Baskerville*, *W. B. Clarke*, *P. J. Hill*, *W. V. Parker* and *C. C. Richtmeyer*.

*N. A. Court, *College Geometry*, p. 73.
†*Ibid.*, p. 86.

No. 470. Proposed by *E. P. Starke*, Rutgers University.

Find a 3-digit number in the scale of 11, which requires the same digits in reverse order when written in the scale of 7.

Solution by *Tryphena Howard*, Western Kentucky State Teachers college.

Let x, y, z be the three digits. Then by hypothesis

$$121x+11y+z=x+7y+49z \qquad \text{or}$$

$$y=12z-30x, \tag{1}$$

in which the letters must represent positive integers less than 7. Evidently y is a multiple of 6. If y is 0 or 6, (1) gives

$$2z=5x \quad \text{or} \quad 2z=5x+1,$$

respectively. Thus the only possible solutions are: $x=1$, $y=6$, $z=3$ and $x=2$, $y=0$, $z=5$.

Also solved by *M. I. Chernofsky* and *George A. Yanosik*.

PROPOSALS

No. 492. Proposed by *Walter B. Clarke*, San Jose, California.

If Δ is the area of the triangle ABC, show that

$$abc(a \cos A+b \cos B+c \cos C)=8\Delta^2.$$

No. 493. Proposed by *E. Hoff*,

If Δ is the area of the triangle ABC, and R its circumradius, show that

$$a \cos^3 A+b \cos^3 B+c \cos^3 C=\Delta(1-4 \cos A \cos B \cos C)/R.$$

No. 494. Proposed by *V. Thébault*, Tennie, Sarthe, France.

Find the smallest possible base of a system of numeration in which the three-digit number 777 is a perfect fourth power.

No. 495. Proposed by *J. P. Wagman*, Washington, D. C.

Let $A=(3\sqrt{3}+5)^{2n+1}$, and let F be the fractional part of A. Show that $2AF=4^{n+1}$.

No. 496. Proposed by *D. L. MacKay*, Evander Childs High School, New York City.

The definition of regular polyhedrons gives three requirements: (a) faces regular polygons, (b) faces congruent, (c) polyhedral angles congruent. Give illustrations of polyhedrons possessing each pair of these requirements but not the third.

No. 497. Proposed by *E. P. Starke*, Rutgers University.

Given any real number N_0. If $N_{j+1} = \cos N_j$, show that the limit of N_j as $j \to \infty$ is a fixed number independent of N_0, and find an approximation of its value.

From Emeritus Professor G. A. Bliss of the University of Chicago on November 25, came the following:

A recent message received through the American Red Cross announces the death in Germany on July 5, 1942 of Professor Oskar Bolza at the age of 85. He was a Reader in Mathematics at Johns Hopkins University in 1888-9, Associate at Clark University 1889-93, Associate Professor at the University of Chicago 1893-4, and Professor 1894-1910. For many years past he has been Nonresident Professor living in Freiburg in B.

Bibliography and Reviews

Edited by
H. A. Simmons and P. K. Smith

Plane and Spherical Trigonometry. By Frank A. Rickey and J. P. Cole. The Dryden Press, New York, 1942. x+209 pages. $2.25.

The authors state in the preface: "The emphasis is placed upon *method* rather than upon rule, upon *remembering* formulas as consequences of the fundamentals rather than upon assigned memory work". The first ten pages of the text contain a review of simple plane geometry, including figures, names and definitions, which will be especially useful to adults using the book for a "refresher course" or to any students whose knowledge of plane geometry $\rightarrow 0$. But is plane geometry not so well known as compass bearings and limits

$$\left(\text{for example, } \lim_{\Theta\to 0} \frac{\Theta}{\sin\Theta},\ \lim_{\Theta\to 0} \frac{\Theta}{\tan\Theta}\right),$$

which are introduced without explanation later on?

The Preface of about two pages is followed by a Table of Contents of three and a half pages. The tenth chapter of Plane Trigonometry ends on page 128. Then come two chapters of Spherical Trigonometry, the first of them being named "Useful Facts About Solid Figures". The Appendix occupies pages 180-191, and includes discussions of (a) the accuracy of computed results and (b) common logarithms. (Why not refer the student to the Appendix when he needs to use the laws of logarithms in the text?) The usual tables (to four places) occupy pages 193-207. The book ends with an Index of two pages that leads one firmly from "abscissa" to "zone". Summaries of formulas are given at the ends of several chapters. And a good set of mixed review exercises in plane trigonometry is found at the end of the chapter on "Complex Numbers and De Moivre's Theorem", but why it should be a part of that chapter is not clear, even though that is the last chapter on plane trigonometry.

The use of the slide rule is suggested, and the accuracy thereof is well stated. Angles of elevation and depression are handled successfully. There are particularly good drawings for a problem on mountains and mile posts and angles of depression (with two different solutions offered) and for a practical problem on *surveying under difficulties.*

Vectors are carefully introduced. Simple, interesting problems on airplane flying and anti-aircraft firing are found. The mil is introduced. Linear speed and angular velocity appear. On page 50 the formula

$$\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=d$$

is derived and illustrated by problems. Then on page 75, after one has had time to forget this formula, it is used rather cleverly to find $\cos(\Theta+\Phi)$. One section and a set of exercises deal with loci in polar coordinates.

The present reviewer likes the notes on the distinction between $\sin^2\alpha$ and $\sin\alpha^2$, and between $\cos(\Theta+\Phi)$ and $\cos\Theta+\cos\Phi$, as well as the treatment of graphs of trigonometric functions (including "adding of ordinates") and the method of handling inverse functions. There are some very good sets of problems of different types, in-

cluding one example "for those who want a mettle tester". The irrigation problems will be rather unpleasant; and the drawing of what is presumably an angel cake pan (figure 90) leaves something to be desired on the subject of whether the *inside* is *outside*.

Some errors are noted: among them A for α and B for β (once each), $2n\pi+\pi$ for $2n\pi\pm\pi$(p. 115); $\text{Tan}^{-1}-3/2$ for $\text{Tan}^{-1}(-3/2)$ (p. 125); $180°-61°33'=18°27'$ (p. 166); and the failure of the lettering of Figure 111 to agree with the text.

Wellesley College. MARION E. STARK.

Basic Mathematics. By William Betz. Ginn and Company, Boston, 1942. x+502 pages. $1.48.

The purpose of this text is to offer "a combination of basic units in arithmetic, informal geometry, elementary algebra, and numerical trigonometry." These units are intended to afford the student an opportunity "to master as quickly as possible the concepts, principles, and processes which constitute the permanent framework of elementary mathematics." The author recommends the text for (1) those students in the upper years of the high school who, whether or not they have had the previous courses in high school mathematics, desire to refresh and to strengthen their knowledge of the fundamentals, and yet may not be able to devote two or more years to the regular courses and (2) those students who have not had the requisite training in arithmetic and informal geometry.

The work of the text is begun with an inventory test in the fundamental processes of arithmetic. The test is followed by the description of a set of important habits to be kept in mind in the study of mathematics and by numerous practice exercises intended to assure mastery of the fundamentals. Each chapter of the book is concluded with a summary and a test. Skills are strengthened by repetition in succeeding chapters.

A treatment of common fractions, decimal fractions, and percentage occupies the next three chapters of ninety pages. Inventory tests appear at the beginning of each chapter in the book. The processes with fractions are explained with diagrams. Considerable space is given to verbal problems and a set of seven definite steps to follow in problem solving is suggested as an aid to the student.

The second part of the book, consisting of eight chapters, is devoted to geometry. The treatment is that commonly found in junior high school textbooks. The sequence of topics includes measurement, the metric system, scale drawing, use of the compasses, the circle, etc. The "mil system," the "radian system," and the "sexagesimal system" are considered in the measurement of angles. A number of simple geometric relationships are deduced.

The third part of the book deals with elementary algebra. Successive chapters include an introduction to the fundamental processes through formulas, statistical and function graphs, equations and verbal problems. Verbal problems are given considerable prominence. The treatment of the fundamental processes is rather limited. In multiplication the multiplier is limited to a monomial, and in division the divisor is limited to a monomial. The meanings of concepts and practice exercises are emphasized.

The last part is a development of trigonometry in four chapters. The material is introduced with indirect measurement and the approximate character of measurements. A brief description of some measuring instruments, the hypotenuse rule, square root, and radicals follow. A table of squares and square roots is discussed. The *tangent ratio*, the *sine ratio*, and the *cosine ratio* are defined and a four-place table of these trigonometric ratios for each degree from zero to ninety degrees inclusive is presented. The use of the table is made clear through the solution of practical problems. Inter-

polation is used with angles expressed to the nearest minute and to the nearest tenth of a degree. The last chapter is devoted to the theory and use of common logarithms and the slide rule.

The text is concluded with one section of review exercises and problems in four parts corresponding to the four parts into which the book is divided, and a final section on exercises in arithmetic.

This book is carefully written, excellently planned, attractive in presentation, and is to be recommended in its field.

Northern Illinois State Teachers College. EUGENE W. HELLMICH.

Manual of Mathematics for Students of Agriculture. By Fred Robertson. The Dryden Press, Inc., New York, 1941. vi+335 pages. $2.50.

The author states that this text is intended for use in agriculture in standard colleges and universities by freshman and sophomore students, who have completed at least one year of high school algebra. It is an outgrowth of courses in mathematics given for students in agriculture at the Iowa State College; and full college credit is given for successful completion of the course.

The book is divided into two parts plus an appendix. Part I contains all the essential topics of an intermediate course in algebra and has, in addition, sections on land location, interest and annuities, and statistics. Part II is an abridged treatment of plane trigonometry, the subject matter of which is adequate for students of agriculture. The appendix consists of a section on the slide rule, a table of equivalent weights and measures, and a list of formulas and constants.

A definite and rather successful effort has been made to select an ample list of problems of an agricultural nature. Answers are given to all problems. Space is provided after each problem for its solution, a feature which should make the book useful for reference in later work. Work-sheets are included to be completed, detached, and turned in to the instructor for scoring. For obvious reasons the student is expected to start the course with a new book.

While there has been a careful and adequate selection of subject matter the reviewer feels that the book is inadequate in amount of explanatory material and examples. It is his opinion that the average student would have difficulty in self-study. To illustrate this point, the following is the complete text of the ambiguous case of oblique triangles:

> "Case II. Given two sides and an angle opposite one of them.
>
> This is the so-called ambiguous case. There are analytic criteria to determine the number of solutions. However, a drawing to scale and a computation using the sine law is adequate to handle most cases.
>
> The sine law applies immediately to cases I and II."

No diagrams and no examples amplify these brief remarks.

The reviewer also wishes to raise objection to the method of treatment of the important subject of fractions. On page 23 there are *formulas* for the addition and subtraction of fractions, after which there is a list of 24 problems on this subject. On page 31 the problem of reduction of fractions to the lowest common denominator is considered, but no mention is made of the application of this process to the addition and subtraction of fractions; and no problems are provided for this purpose.

If the instructor is willing to amplify the explanatory material to a considerable extent, and if he is willing to overlook certain defects, such as the one cited in the pre-

ceding paragraph, the reviewer feels that an effective course in mathematics for students of agriculture could be given with this book as a text.

The University of California. A. C. BURDETTE.

TEACHING AIDS FOR THE WAR-TIME PROGRAM

As a part of its contribution to the Victory Corps Program, the New Jersey State Teachers College, Upper Montclair, N. J., offers the services of its War Information Center and Teaching Aids Service, both departments of the College Library.

The College was designated by the School and College Civilian Morale Service of the U. S. Office of Education as one of the three Key War Information Centers in New Jersey colleges. The Information Center is on the free mailing lists of 129 organizations, distributed as follows:

Government agencies, Federal, State and Local	37
Propaganda and information services of the United Nations	13
Information services of American groups of foreign origin	5
Associations for social and economic betterment, post-war planning, etc.	49
Commercial organizations publishing informational and morale-building materials	13
Miscellaneous	12

These materials, as well as books, pamphlets, etc. from the Library of the College, are classified and available for use at all times. In addition, the Library has published two selected lists, with supplements, on Civilian Morale, and Post-War Planning and the Schools. (5c. each).

The Teaching Aids Service has been engaged since 1938 in collecting materials and information useful to teachers in junior and senior high schools. Many of these materials are also of value in the elementary field. The catalog of the Service now includes more than 11,000 entries, under 1500 subjects. Continuous research adds data daily.

A number of lists of Visual and Teaching Aids, mentioned in detail below, and based on this catalog, are now available to curriculum laboratories, state and city boards of education, libraries, museums, and individual teachers throughout the country. Since they are up-to-date, the materials in these lists and in the files fit into the War-Time Program outlined in the *Bulletin of the National Association of Secondary School Principals, Vol. 26, No. 108, October, 1942.* On pp. 74-75 of this Bulletin, the objectives of the High School Corps Program are defined as:

1. Guidance into critical services and occupations.
2. War-time citizenship.
3. Physical fitness.
4. Military drill.
5. Competence in science and mathematics.
6. Pre-flight training in aeronautics.
7. Pre-induction training for critical occupations.
8. Community services.

The publications of the Teaching Aids Service fit into the various phases of this program as indicated below.

A. HEALTH EDUCATION. 1941. 25c.
Guidance, Physical fitness, Pre-induction training, Community services.

B. SAFETY EDUCATION. 1941. 15c.
Guidance, War-time citizenship, Pre-induction training, Physical fitness, Community services.

C. RECREATION. In preparation.
Guidance, Physical fitness, Community services.

D. JOURNEYS INTO SCIENCE. In preparation.
Guidance, Physical fitness, Competence in science, War-time citizenship, Community services.

E. PROBLEMS OF AMERICAN DEMOCRACY. 1941. 50c.
Guidance, War-time citizenship, Community services.

F. VISUAL AIDS IN THE REALM OF BIOLOGY. 1941. 50c.
Guidance, Physical fitness, Competence in science, Community services.

G. PANAMERICANA. 2 volumes. 1940-1942. $1.00.
War-time citizenship.

H. MUSIC IN THE JUNIOR AND SENIOR HIGH SCHOOL. 1941. 25c.
Guidance, War-time citizenship, Community services.

J. MATHEMATICS. 1942. 25c.
Guidance, Competence in mathematics, Pre-flight training in aeronautics, Pre-induction training in critical occupations.

K. FLYING AND WEATHER. 1942. 50c.
Guidance, War-time citizenship, Competence in science and mathematics, Pre-flight training, Pre-induction training.

L. BUSINESS EDUCATION. In preparation.
Guidance, War-time citizenship.

M. AIDS FOR THE FRENCH TEACHER, comp. by Lili Heimers. Pub. by E. G. Stechert & Co., 31 East 10th St., N. Y. 1938. 50c.
Guidance, War-time citizenship.

N. AIDS FOR THE SPANISH TEACHER, comp. by Lili Heimers. Pub. by G. E. Stechert. 1940. 50c.
Guidance, War-time citizenship.

With the exception of the last two items, these are all available from

TEACHING AIDS SERVICE
N. J. STATE TEACHERS COLLEGE
UPPER MONTCLAIR, N. J.

With the same exceptions, all are free, upon application, to librarians in the public schools of New Jersey.